Extreme Value Theory and Applications

Nabil KAZI-TANI - ISFA - Université de Lyon 1

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Introduction

- Extreme (from Latin *exter, exterus*, being on the outside) : Exceeding the ordinary, usual, or expected.The mathematical meaning and modelling of extreme risks requires specific tools.
- We will deal here with the univariate EVT in the absence of temporal dependence.
- As part of EVT, we study the **asymptotic behavior** of :
  - The Sum (in the form of reminders).
  - The Maximum, especially through the Fisher-Tippett theorem.
  - The Excess $X - u$, knowing that $X > u$, for a sufficiently large threshold u, by the theorem of Pickands, Balkema, de Haan.

LLN and CLT

Let $X_1, \ldots, X_n$ be i.i.d. r.r.v. (real random variables) and $S_n = \sum_{i=1}^{n} X_i$.

Law of Large Numbers

If $E(|X_1|) < +\infty$, then S_n/n converges a.s. (almost surely) to $E(X_1)$. In other words, $(\frac{S_n}{n} - E(X_1))$ converges a.s. to 0.

Central Limit Theorem

If $E(|X_1|^2) < +\infty$, then $\frac{S_n - nE(X_1)}{\sqrt{nV(X_1)}}$ converges in law to a r.r.v. of standard normal distribution $N(0, 1)$.

This explains the importance of the gaussian distribution.

Asymptotic behavior of the Max

Notations

- Let $X_1, \ldots, X_n$ be i.i.d. r.r.v. with cumulative function F and let $M_n = \max(X_1, \ldots, X_n)$.
- Let $x_F =$ maximal value taken by $X = sup\{x \in R / F(x) < 1\} \leq +\infty$.

- Extreme value distributions are the **limiting distributions** for the minimum or the maximum of a very large collection of random observations from the same arbitrary distribution.
- Only a few models are needed !

Asymptotic behavior of the Max

Problem

Can we find two real positive sequences (a_n) and (b_n) such that $\frac{M_n - a_n}{b_n}$ converges in law to a non degenerate real random variable ?

Proposition

- M_n converges a.s to x_F.

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Asymptotic behavior of the Max

Example with a random number of losses

- Let N be a discrete r.v with $P(N \leq x) = F_N(x)$.
- We can compute $F_{M_N}(x) = P(M_N \leq x)$ as

The maximum loss

$$P(M_N \leq x) = E([F(x)]^N)$$

If we can specify the distribution of the frequency and severity of losses, we can easily obtain the exact distribution of the maximum loss.

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Asymptotic behavior of the Max

Definitions

- **standard GEV law** : its d.f. H_ξ (where $\xi \in R$ is a shape parameter) is defined for all x such that $1 + \xi x > 0$ by :
  - $H_\xi(x) = exp(-(1 + \xi x)^{-1/\xi})$, Case $\xi \neq 0$.
  - $H_\xi(x) = exp(-exp(-x))$, Case $\xi = 0$
  - Note : if $\xi = 0$ obtained by continuous extension.

Fisher-Tippett Theorem

If there exist scaling constants a_n and $b_n > 0$ such that

$$\mathbf{P}\left[\frac{max_i X_i - a_n}{b_n} \leq x\right] \overset{n\to\infty}{\to} F(x)$$

where F is a nondegenerate d.f. then $F = H_\xi$.

Definitions

The 3 following limit laws Ψ_α, Λ and Φ_α are defined as special cases of H_ξ :

- Case $\xi < 0$ (Weibull) : $\Psi(x) = H_\xi(-(x+1)/\xi)$.
- Case $\xi = 0$ (Gumbel) : $\Lambda(x) = H_0(x)$.
- Case $\xi > 0$ (Fréchet) : $\Phi(x) = H_\xi((x-1)/\xi)$.

Asymptotic behavior of the Max

Definition, Maximum Domain of Attraction

- The X_i are i.i.d. with c.f. F. Then $MDA(G)$ (for a c.f. G) is defined as the set of laws with c.f. F such that there exist two real sequences (a_n) and $(b_n) > 0$ such that $(M_n - a_n)/b_n$ converges in law to a r.v with c.f. G.
- We shall distinguish 3 sorts of domains of attraction of the max : $MDA(\Psi)$ (Weibull), $MDA(\Lambda)$ (Gumbel), $MDA(\Phi)$ (Fréchet).

Examples

- Examples of laws in $MDA(\Phi)$: Pareto, Cauchy.
- Example of law in $MDA(\Psi)$: beta law (the uniform law on $[0,1]$ is a special case of the beta law).
- Examples of laws in $MDA(\Lambda)$: normal, log-normal, gamma. Note : The exponential law is a special case of the gamma law.

Characterization of $MDA(\Lambda)$ (Gumbel case)

Theorem

If F is a C^2 function, we have the simple condition : $F \in MDA(\Lambda)$ if and only if $\lim_{x \to \infty} \frac{(1-F(x))F''(x)}{F'(x)^2} = -1$

Exploitation of the Fisher-Tippett theorem in practice

In practice, the maximum of n variables $X_1, ..., X_n$ constitutes only a unique observation, which makes the law of this single observation delicate to estimate.
Let us consider a sample $X_1, ..., X_n$ of i.i.d. r.v. and note $Y = max(X_1, ..., X_n)$.
If we can have a sample of maxima $Y_1, ..., Y_m$, classic methods of parameter estimation would be possible : likelihood maximum for example.

The study of the maximum has been historically the first method to study the extreme phenomena. Nevertheless, in risk management, we are also interested in the law of the excesses, i.e. the law of $X - u$ knowing $X > u$, for a threshold u big enough.

Example

$[(X - u)|X > u] = X$ in law, if and only if X is exponentially distributed.

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Generalized Pareto Distribution *GPD*

Definition

- The $GPD(\xi)$ law is defined by its c.f. G_ξ :
  - In case $\xi \neq 0$: $G_\xi(x) = 1 - (1 + \xi x)^{-1/\xi}$.
  - In case $\xi = 0$: $G_\xi(x) = 1 - exp(-x)$, i.e. the exponential law of parameter 1.

  The support of G_ξ is $[0; +\infty[$ for $\xi \geq 0$, $]0; -1/\xi[$ for $\xi < 0$.
- The $GPD(\xi, \mu, \sigma)$ law, with $\mu \in R$ and $\sigma > 0$, is defined by its c.f. $G_{\xi,\mu,\sigma}(x) = G_\xi((x - \mu)/\sigma)$, the support of $G_{\xi,\mu,\sigma}$ being $[\mu; +\infty[$ for $\xi \geq 0$, $]\mu; \mu - \sigma/\xi[$ for $\xi < 0$.

Generalized Pareto Distribution *GPD*

Notation

- Let $F_u(x) = P(X - u \leq x | X > u)$.
- Let $e(u) = E(X - u | X > u)$

Property

The class of laws $GPD(\xi, \mu, \sigma)$ is stable by truncation to the left.
If X is a r.v. of law $GPD(\xi, \mu, \sigma)$ with c.f. F, and u is a threshold inside the support of X, then, $X - u | X > u$ follows a $GDP(\xi, 0, \sigma + \xi(u - \mu))$ law.

Theorem of Pickands, Balkema, de Haan

Theorem

$F \in MDA(H_\xi)$ if and only if :

$$\lim_{u \to x_F} \sup_{x \in [0; x_F - u[} |F_u(x) - G_{\xi,0,\sigma(u)}(x)| = 0$$

where H_ξ is the law $GEV(\xi)$, $G_{\xi,0,\sigma(u)}$ is a c.f. of the $GPD(\xi, 0, \sigma(u))$ law and $\sigma(.)$ is a positive function.

Using the Characterization of $MDA(\Phi)$

Applications

Estimating the shape parameter (Hill estimator) :

$$\hat{\alpha}(k) = \frac{1}{k}\sum_{i=1}^{k-1}(\log(X_{n-i+1,n}) - \log(X_{n-k+1,n}))$$

In Practice

- Hill estimator is widely used in practice.
- There is a difficulty in the choice of the parameter k.
- If k is small, $\hat{\alpha}(k)$ uses a small number of observations : it has a large variance.
- If k is large, the variable $X_{n-k+1,n}$ is small, we are outside of the zone of approximation of the survival function by a power function : the estimator has a large bias.

Characterizing the MDA

Let $U(t) := 1 - F(1 - \frac{1}{t})$. F is in $MDA(H_\xi)$ if and only if there exist a function ψ such that

$$\lim_{t \to x_F} \frac{U(tx) - U(t)}{\psi(U(t))} = h_\xi(x)$$

où $h_\xi(x) = \frac{x^\xi - 1}{\xi}$ if $\xi \neq 0$ and $h_\xi(x) = \ln(x)$ if $\xi = 0$.

From this we can get that

$$\lim_{x \to +\infty} \frac{E[X - x | X > x]}{\psi(x)} = \frac{1}{1 - \xi}.$$

Pure premium in reinsurance

Pure premium in reinsurance :
$\Pi(R) = E[(X-R)^+] = E[X-R|X>R]P(X>R)$.

For R big enough and $0<\xi<1$, we approximate

$$\Pi(R) \approx \frac{\psi(R)}{1-\xi} P(X>R).$$

If $F \in MDA(H_\xi)$ with $\xi>0$, then $\psi(R)=\xi R$. If we choose $R = X_{(n-k:n)}$:

$$\Pi(R) \approx \frac{\hat{\xi}}{1-\hat{\xi}} \frac{k}{n} X_{(n-k:n)}.$$

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