

# Vision conditionnelle du monde dans les stress tests et révision des hypothèses actuarielles

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Journées de Deauville & Formation Actuaire Expert ERM

# Flashback: Examen ERM 2012



# Flashback: ERM 2012



# Question d'un ERM 2012 anonyme



# SCR cat vie

- Choc: accroissement instantané (non permanent) de 0,15% du taux de mortalité (pendant 12 mois)
- Si décès de l'assuré génère une rente de conjoint survivant ou une rente éducation,

le choc s'applique-t-il aux rentes (avec du coup un effet atténuation du choc)?

# Quelques éléments de réponse

- Choc calibré sur risque pandémie.
- Risque nucléaire mentionné.
- Présence de l'expression « par exemple »
- *The instantaneous increase in mortality rate shall only apply to those policies for which an increase in mortality rates which are used to reflect the mortality experience in the following 12 months leads to an increase in technical provisions.*

# Plus généralement, dans un stress test,

- Comment simuler le « monde conditionnel »?
- Réponse en proba: conditionner sachant que  $X=x$  n'est pas pertinent.
- Il est plus pertinent de conditionner par  $X$  est pire que  $x$ .
- Lien avec allocation de capital économique avec méthode d'Euler:
  - Fait du sens avec TVaR
  - Pas avec VaR

# Exemple: Inconvénient de la Co-VaR

- Co-VaR: proposée par Fed il y a quelques années pour mesurer la sensibilité d'une banque au risque systémique
- Idée: sachant que le marché (banques) est dans un état désastreux, que devient la VaR conditionnelle de notre banque?
- Etat désastreux: représenté par le marché est (exactement!) à sa VaR

## How to measure sensitivity to systemic risk?

To measure what would happen if the hedge fund market or if the investment banking sector gets into trouble, several approaches may be proposed, including:

- if there are a few big players, remove one big player from the market and see what happens (Lehmann-type scenario): some institutions may have the feeling to be hedged against some risk despite the risk that the counterpart disappears. Example: portfolio of long and short positions into interest rate swaps (with maturities 30 years, 29 years, and so on...). The regulator needs to be able to determine what would happen...
- In addition to the institution's "stand-alone" Value-at-Risk, ask all institutions to report their exposure towards systemic risk. The Federal reserve proposed to measure this by the means of the so-called Co-Value-at-Risk (Co-VaR) or Co-Expected Shortfall (Co-ES).

## Co-Value-at-Risk

- For  $\alpha \in (0, 1)$ , define the  $\alpha$ -Co-Value-at-Risk of  $Y$  given  $X$  as

$$\text{CoVaR}_\alpha(Y | X) = \text{VaR}_\alpha(Y | X = \text{VaR}_\alpha(X)).$$

- Using the copula approach, with  $U = F_X(X)$  and  $V = F_Y(Y)$ , we have

$$\text{CoVaR}_\alpha(Y | X) = F_Y^{-1} \left( C_{2|1}^{-1}(\alpha | \alpha) \right),$$

where  $C$  is the copula of  $X$  and  $Y$  and where

$$C_{2|1}(v | u) = \frac{\partial C}{\partial u}(u, v)$$

is the conditional distribution of  $V$  given that  $U = u$ .

- The idea is the following: if  $Y$  corresponds to wealth of institution A and if  $X$  corresponds to a market index,  $\text{CoVaR}_\alpha(Y | X)$  is the conditional  $\alpha$ -VaR of institution A given that the market index is at its  $\alpha$ -VaR: if  $\alpha$  is small, this corresponds to a conditional VaR in a market distress scenario.

# Co-Value-at-Risk

- For  $\alpha \in (0, 1)$ , define the  $\alpha$ -Co-Value-at-Risk of  $Y$  given  $X$  as

$$\text{CoVaR}_\alpha(Y | X) = \text{VaR}_\alpha(Y | X = \text{VaR}_\alpha(X)).$$

- The goal is to take into account systemic risk: if institution  $A$  is hedged against some risks using the market, then  $X$  and  $Y$  should be positively correlated and we expect Co-VaR to take this into account by increasing the risk capital needed to control it (in comparison to the case where  $X$  and  $Y$  are independent).
- Problem: this is not the case!

## Co-Value-at-Risk: Clayton copula

- The Clayton copula with parameter  $\theta > 0$  is given by

$$C_{\theta}(u, v) = \left( u^{-\theta} + v^{-\theta} - 1 \right)^{-\frac{1}{\theta}}.$$

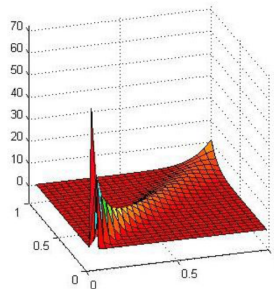
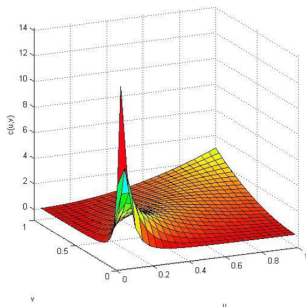
- Keeping in mind that

$$\text{CoVaR}_{\alpha}(Y | X) = F_Y^{-1} \left( C_{2|1}^{-1}(\alpha | \alpha) \right),$$

we can compute explicitly  $C_{2|1}^{-1}(\alpha | \alpha)$  for the Clayton copula:

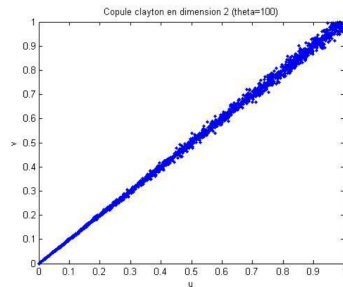
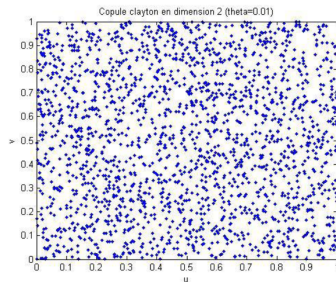
$$C_{2|1}^{-1}(\alpha | \alpha) = \left[ \alpha^{-\theta} \left( \alpha^{-\frac{\theta}{\theta+1}} - 1 \right) + 1 \right]^{-\frac{1}{\theta}}.$$

# Clayton copula



Density of the Clayton copula,  $\theta = 2$  (left) and  $\theta = 100$  (right).  
Correlation increases with  $\theta$ .

# Clayton copula



Simulations of the Clayton copula,  $\theta = 0.01$  (left, close to independence) and  $\theta = 100$  (right, close to comonotonicity).

## Co-Value-at-Risk: Clayton copula

- $$\text{CoVaR}_\alpha(Y | X) = F_Y^{-1} \left( C_{2|1}^{-1}(\alpha | \alpha) \right),$$

- We can compute explicitly  $C_{2|1}^{-1}(\alpha | \alpha)$  for the Clayton copula:

$$C_{2|1}^{-1}(\alpha | \alpha) = \left[ \alpha^{-\theta} \left( \alpha^{-\frac{\theta}{\theta+1}} - 1 \right) + 1 \right]^{-\frac{1}{\theta}}.$$

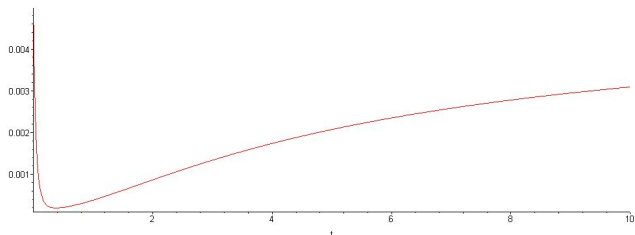
- Problem: as  $\theta \rightarrow 0$  and as  $\theta \rightarrow +\infty$ , we have

$$C_{2|1}^{-1}(\alpha | \alpha) \rightarrow \alpha.$$

- This means that if  $Y$  is independent from the market index, or if  $Y$  is perfectly correlated with the market index  $X$ , then

$$\text{CoVaR}_\alpha(Y | X) = \text{VaR}_\alpha(Y).$$

## Co-Value-at-Risk: Clayton copula



$\beta$  such that  $CoVaR_{\alpha}(Y | X) = VaR_{\beta}(Y)$  as a function of correlation parameter  $\theta$  for the Clayton copula. The lower  $\beta$ , the higher the required capital.

Conclusion: one must condition w.r.t.  $X < VaR_{\alpha'}(X)$  instead of  $X = VaR_{\alpha}(X)$ .

# Et pour des stress tests:

- Faut-il / Devrait-on / Peut-on réestimer le SCR ?
- Probabilités et périodes de retour réévaluées:
  - 25 déc. 1999: Période de retour d'un événement comme Lothar dans un certain modèle CAT = 120 ans
  - Le lendemain soir: on ne sait plus!
  - Janvier 2000: Période de retour d'un événement comme Lothar = 70 ans
  - A. Mornet, M. Luzi, T. Opitz, S. Loisel, **Construction of an Index that links Wind Speeds and Strong Claim Rate of Insurers after a Storm in France**, *Risk Analysis* (2015).
- Pour d'autres risques: retour à la moyenne -> réduction du risque en univers choqué
- Loss absorbing capacity!!!
- Article en préparation avec F. Borel-Mathurin (ACPR) et J. Segers (Louvain)

# Partie 2: Révision des hypothèses actuarielles

- Supposons qu'on observe une déviation de la fréquence de sinistres ou d'autres événements par rapport aux hypothèses actuarielles.
- D'un point de vue statistique, comment réagir le plus vite possible sans faire trop de fausses alarmes?

# Online monitoring of actuarial assumptions

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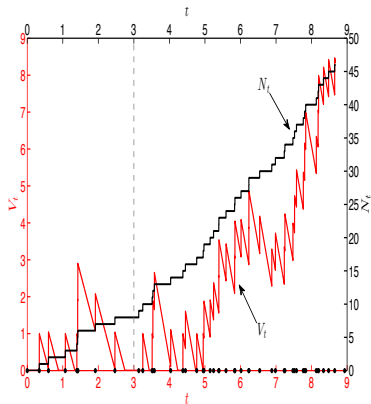
with N. El Karoui and Y. Salhi



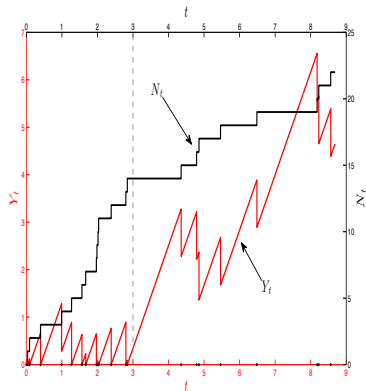
Yahia did not like ruin theory...



# Typical paths with change of regime at date 3



(a) Processes  $N$  and  $V_t$



(b) Processes  $N$  and  $Y_t$

**Figure:** Sample paths, for  $\rho = 1.5$ , of the cusum processes  $N$ ,  $V_t^\rho$  (left) and  $N$ ,  $Y_t^\rho$  for  $\rho = 0.5$  (right).



# Bayesian setup for random change-point

## Brownian framework with abrupt change in the drift

- ▶ Based on the conditional distribution of the time of change,
- ▶ Formulated as an optimal stopping problem
- ▶ Page(1954), Shiryaev(1963), Roberts(1966), Beibel(1988), Moustakides (2004), and many others...

## Poisson framework with abrupt change in intensity

- ▶ Based on the conditional distribution of the time of change, with exponential or geometric prior distribution
- ▶ More recent studies : Gal (1971), Gapeev (2005), Bayraktar (2005, 2006), Dayanik (2006) for compound Poisson, Peskir, Shyriaev(2009) and others

# MATHEMATICAL SETTINGS

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We consider a portfolio of insured population:

- Let  $N = (N_t)_{t \geq 0}$  be a **counting process** indicating the deaths of policyholders and  $\lambda = (\lambda_t)_{t \geq 0}$  its **intensity**.
- The counting process  $N_t$ , is **available sequentially** through the filtration  $\mathcal{F}_t = \sigma\{N_s, 0 < s \leq t\}$ .
- We suppose that the insurance company relies on a **Cox-like** model to project her own experienced mortality:

$$\lambda_t = \underline{\rho} \lambda_t^0,$$

- $\lambda_t^0$  is a **reference intensity** and  $\underline{\rho}$  is a positive parameter.
- $\lambda^0$  is considered deterministic and may refer whether to a projection of national population/best estimate...

Model risk/parameter uncertainty: **Change-point**

$$\lambda_t = \mathbf{1}_{\{t < \theta\}} \underline{\rho} \lambda_t^0 + \mathbf{1}_{\{t \geq \theta\}} \bar{\rho} \lambda_t^0.$$

Without loss of generality we can assume that  $\underline{\rho} = 1$  and let  $\rho = \bar{\rho} > 1$ .

# PROBABILISTIC FORMULATION

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Let  $\mathbb{P}_\theta$  (resp.  $\mathbb{E}_\theta[\cdot]$ ) be the probability measure (resp. expectation) induced when the change takes place at time  $\theta$

## Example

- For  $\theta = 0$ , the process is *out-of-control*
- For  $\theta = \infty$ , the process is *in-control*

**Detect the change-point  $\theta$  as quick as possible while avoiding false alarms**

## OPTIMALITY CRITERIA, LORDEN (1971)-LIKE

- The detection delay  $\mathbb{E}_\theta \left[ (N_\tau - N_\theta)^+ \mid \mathcal{F}_\theta \right]$
- The frequency of false alarm  $\mathbb{E}_\infty [N_\tau]$

# OPTIMIZATION PROBLEM

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## OPTIMIZATION PROBLEM

Find  $\tau^*$  such that  $C(\tau^*) = \inf_{\tau} \sup_{\theta \in [0, \infty]} \text{ess sup } \mathbb{E}_{\theta} \left[ (N_{\tau} - N_{\theta})^+ \middle| \mathcal{F}_{\theta} \right]$   
subject to  $\mathbb{E}_{\infty}[N_{\tau}] = \omega$ .

## ASSUMPTION

- 1  $\int_0^t \lambda_s ds < \infty, \quad \mathbb{P}_{\infty}, \mathbb{P}_0\text{-a.s.}$
- 2  $N_{\infty} = \infty \quad \mathbb{P}_{\infty}, \mathbb{P}_0\text{-a.s.}$

# OPTIMALITY OF THE CUSUM PROCEDURE (1/7)

Let the Radon-Nikodym density of  $\mathbb{P}_0$  with respect to  $\mathbb{P}_\infty$  be defined as

$$\frac{d\mathbb{P}_0}{d\mathbb{P}_\infty} \Big|_{\mathcal{F}_t} = \exp U_t,$$

where  $U_t = \log(\rho)N_t + (1 - \rho) \int_0^t \lambda_s^0 ds$  is the log-likelihood ratio.

Let  $V(x)$  be the CUSUM process; with head-start  $0 \leq x < m$ ; defined as

$$V_t(x) = U_t - (-x) \wedge \underline{U}_t \quad (1)$$

where  $\underline{U}_t$  is the running infimum of  $U$ , i.e.  $\underline{U}_t = \inf_{s \leq t} U_s$ .

The process  $V(x)$  measures the size of the drawup, comparing the present value of the process  $U$  to its historical infimum  $\underline{U}$ .

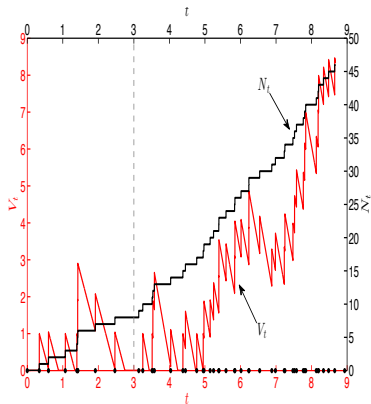
Let  $\tau_m(x)$  be the first hitting time of  $V(x)$  of the barrier  $m$ , i.e.

$$\tau_m(x) = \inf\{t \geq 0, V_t(x) \geq m\}.$$

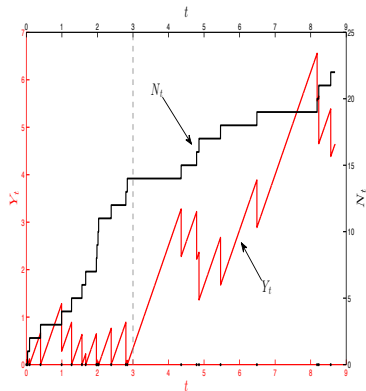
## Theorem

If  $\mathbb{E}_\infty[N_{\tau_m(0)}] = \omega$  then  $\tau_m(0)$  is optimal, i.e.  $\inf_\tau C(\tau) = C(\tau_m(0))$

# Typical paths with change of regime at date 3



(a) Processes  $N$  and  $V_t$



(b) Processes  $N$  and  $Y_t$

**Figure:** Sample paths, for  $\rho = 1.5$ , of the cusum processes  $N$ ,  $V_t^\rho$  (left) and  $N$ ,  $Y_t^\rho$  for  $\rho = 0.5$  (right).

## Performance functions of the V-cusum rule

- ▶ The performance of the cusum stopping is based on

$$\Gamma_t^m(x) = \tilde{\mathbb{E}}_x \left[ \mathbf{1}_{\tau_m^V \geq t} (N_{\tau_m^V} - N_t) \mid \mathcal{F}_t \right] = \tilde{h}_m(V_t(x)) \quad \tilde{\mathbb{P}} \text{ a.s.}$$

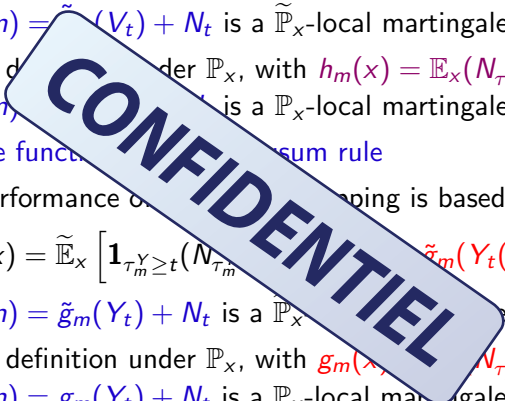
- ▶  $\tilde{H}_t(x, m) = \tilde{h}_m(V_t) + N_t$  is a  $\tilde{\mathbb{P}}_x$ -local martingale on  $[0, \tau_m^V)$ ,
- ▶ Similar definition under  $\mathbb{P}_x$ , with  $h_m(x) = \mathbb{E}_x(N_{\tau_m^V})$  and  $H_t(x, m) = h_m(V_t) + N_t$  is a  $\mathbb{P}_x$ -local martingale on  $[0, \tau_m)$

## Performance functions of the Y-cusum rule

- ▶ The performance of the cusum stopping is based on

$$\Gamma_t^m(x) = \tilde{\mathbb{E}}_x \left[ \mathbf{1}_{\tau_m^Y \geq t} (N_{\tau_m^Y} - N_t) \mid \mathcal{F}_t \right] = \tilde{g}_m(Y_t(x)) \quad \tilde{\mathbb{P}} \text{ a.s.}$$

- ▶  $\tilde{G}_t(x, m) = \tilde{g}_m(Y_t) + N_t$  is a  $\tilde{\mathbb{P}}_x$ -local martingale on  $[0, \tau_m)$
- ▶ Similar definition under  $\mathbb{P}_x$ , with  $g_m(x) = \mathbb{E}_x(N_{\tau_m^Y})$  and  $G_t(x, m) = g_m(Y_t) + N_t$  is a  $\mathbb{P}_x$ -local martingale on  $[0, \tau_m)$



## DETECTION PROCEDURE – ALGORITHM

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- Step 1:** Fix the input parameters: The post-change intensity through the specification of  $\rho$  and the false alarm constraint  $\omega$ .
- Step 2:** Determine the threshold  $m$  as the solution of the equation  $\mathbb{E}_\infty[N_{\tau_m}] = \omega$ .
- Step 3:** For each new observation at time  $t$  compute the value of the CUSUM process  $V$  given by the iterative relation  $V_{t+1} = (V_{t-1} + U_t)^+$ .
- Step 4:** Compare the current value of  $V$  to the threshold  $m$  and stop the procedure once  $V_t \geq m$  and sound an alarm. Hence  $\tau_m(0) = t$ .

## DETECTION PROCEDURE – REAL WORLD (1/4)

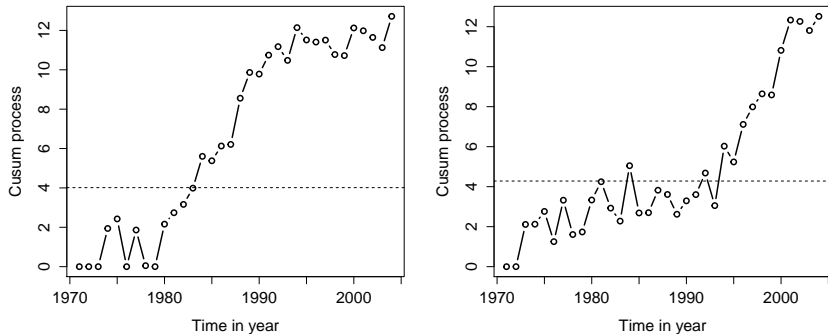
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We consider the CONTINUOUS MORTALITY INVESTIGATION assured lives dataset and England & Wales national population. We split data into two periods:

- We consider the period 1947-1969 as a training period.
- The Cox model is estimated over this period using the MLE.

Hence we monitor the sequentially the dataset over the period 1970-2005 and look for changes on the mortality of assured lives.

## DETECTION PROCEDURE – REAL WORLD (2/4)



**Figure:** Detection scheme for age groups 50 – 59 (right) and 80 – 89 (left). The post-change is set to  $\rho = 15\%$  and the false alarm constraint to  $\omega = 100\bar{\lambda}$ .

## DETECTION PROCEDURE – REAL WORLD (4/4)

Age	$\tau_m$		Observed
	$\rho = 1.50$	$\rho = 1.15$	
50 – 59	1984	1978	1970
60 – 69	1991	1985	1974
70 – 79	1988	1984	1974
80 – 89	1983	1978	1973

**Table:** Detection of mortality change with a post-change ratio of  $\rho = 1.15$  and an average run length (false alarm) constraint of 100. The right column reports the detected change-point using an off-line procedure.

# Monitoring Mortality

Sounding an alarm for the change  $\rho^{\text{Hyp}} \rightarrow \rho^{\text{Targer}}$

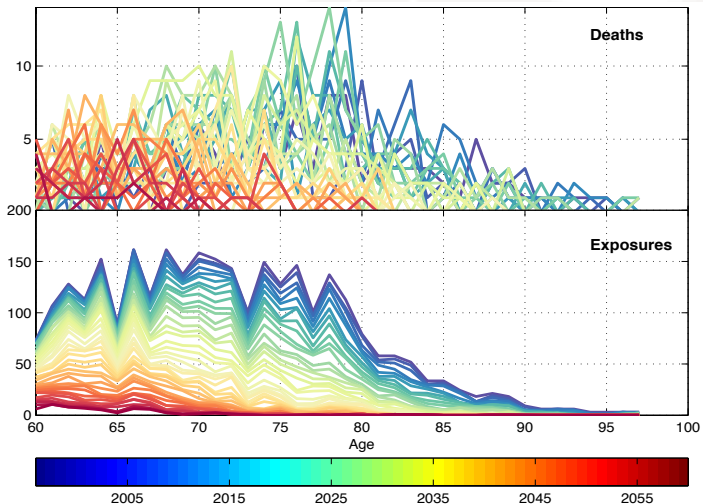
- We simulate deaths on the portfolio with different levels  $\rho^{\text{Targer}} = 95\%, 90\%$  and  $85\%$  s.t.

$$D(x, t) \sim \text{Pois}(\rho^{\text{Targer}} \times L(x, t) \times \mu^{\text{ERM00}}(x, t))$$

- We suppose that *the actuary* made an assumption of  $\rho^{\text{Hyp}} = 100\%$
- We set-up the monitoring/surveillance on the observed deaths and try to detect a change from  $\rho^{\text{Hyp}} = 100\%$  to  $\rho^{\text{Targer}} = 95\%, 80\%$  and  $85\%$  respectively.
- We test different sizes of the portfolio small sized 1000, 5000 and a (relatively) large 10000 and compare the results

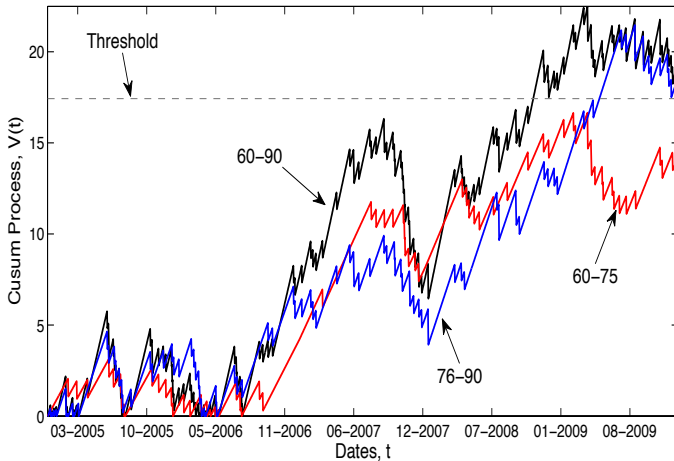
# Monitoring Mortality

Sounding an alarm for the change  $\rho^{\text{Hyp}} \rightarrow \rho^{\text{Targer}}$



# Monitoring Mortality

Sounding an alarm for the change  $\rho^{\text{Hyp}} = 100\% \rightarrow \rho^{\text{Target}} = 95\%$



# Detection Delay

## Impact of Portfolio Size and Age Tranches

Size		1000			5000			10000		
Hyp.	Ages	60-90	60-75	76-90	60-90	60-75	76-90	60-90	60-75	76-90
	deaths	100% → 95%	596	710	498	246	99	107	240	99
100% → 90%		244	320	186	106	55	59	112	55	58
100% → 85%		92	122	100	58	35	36	61	34	36
time	100% → 95%	1086	1130	1120	576	617	422	308	327	212
	100% → 90%	931	1124	947	276	373	241	151	192	127
	100% → 85%	707	980	734	161	247	159	84	127	80



# CONCLUSION

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- The CUSUM detection rule is optimal in the case of non-homogeneous Poisson process with a modified Lorden criterion
- The considered methodology can be extended to the case where the mortality intensity is stochastic
- The "quickness" depends on the size of the change and the initial population

